18 July Anthony Blanc: "Deformations of objects & categories"

Much of the content is in DAG X §5

The cool title of this talk would be

"Moduli of branes & nc spaces"

§0 Let's give some context & motivation.

Mirror symmetry in physics

Witten:

$$\begin{tikzcd} & X \text{ CY manifold} \arrow[dl, squiggly] \arrow[dr, squiggly] & \\ \text{A-model} & & \text{B-model} \end{tikzcd}$$

CFT = conformal field theories

"Def" Y CY manifold is "mirror to X" if

- A-model(X) $\simeq$ B-model(Y)
- B-model(X) $\simeq$ A-model(X)

These CFTs have boundary conditions which live in Fuk(X) & $D^b_{Coh}(X)$
A-model B-model

Kontsevich: introduced a new version

"**Def**" X is "mirror to Y" if

$$\mathrm{Fuk}(X) \simeq D^b_{Coh}(Y) \ \& \ D^b_{Coh}(X) \simeq \mathrm{Fuk}(Y).$$

Costello: ~~these~~ TCFTs ("topological CFT's") are

contains A- & B-models

equivalent to A_∞-categories (w/ extra conditions)

Upshot: the space X is not intrinsic to the physics, only its "shadows" ($\mathrm{Fuk}(X)$, $D^b_{Coh}(X)$) matter for the physics

$\Rightarrow$ "forget X" & focus on these noncommutative "shadows" //

Def (Kontsevich) k = commutative ring

A noncommutative space is a k-linear dg category ($\mathbb{Z}/2$- or $\mathbb{Z}$-graded)

will give definition in a moment

Ex $\mathrm{Fuk}(X)$, $D^b_{Coh}(X)$, rep categories of quivers, matrix factorizations, ...

My primary interest are examples like
$$D^b_{Coh}(X) \quad \& \quad D^b_{qcoh}(X)$$
as they relate (commutative) geometry to noncommutative geometry:

$$\begin{array}{ccc} \mathrm{Var}/_{\mathbb{C}} & \longrightarrow & \mathrm{dgcat}_{\mathbb{C}} \\ X & \longmapsto & D^b_{Coh}(X) \end{array}$$

varieties

There is a functor going the other way, if you enlarge $\mathrm{Var}/_{\mathbb{C}}$ a little:

$$\begin{array}{ccc} \mathrm{dgcat}_{\mathbb{C}} & \longrightarrow & \mathrm{Stacks}/_{\mathbb{C}} \\ \mathcal{E} & \longmapsto & \mathcal{M}_{\mathcal{E}} \end{array}$$

$\mathcal{M}_{\mathcal{E}}$ = moduli stack of compact $\mathcal{E}^{op}$-dg-modules

[Toën-Vaquié, Pandit]

By moving back to commutative geometry, you can often do computations & find invariants for noncommutative spaces.

There is a moduli space of nc-spaces:

$$\underline{\mathrm{dgcat}}^{\mathrm{big}} : \mathrm{CAlg}_k \longrightarrow \mathcal{S}$$

$$A \longmapsto \underline{\mathrm{dgcat}}^{\mathrm{big}}_A = \left(\begin{matrix} \text{big } A\text{-linear} \\ \text{dg categories up} \\ \text{to quasi-iso} \end{matrix} \right)$$

Toën: this gives a unifying perspective on deformation quantization:

ICM 2014 talk

- quantum groups
- skein algebras
- Donaldson-Thomas invariants

} all can be formulated as deformations of E_n-monoidal categories, $n \in \mathbb{N}$

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§1 Dg categories

Def k — commutative ring

A k-linear dg category $\mathcal{C}$ consists of the data of

— a set of objects $\mathrm{Ob}(\mathcal{C}) = \{E, F, \dots\}$

— for any two objects E, F, a cochain complex of k-modules

$$\underline{\mathrm{Map}}_{\mathcal{C}}(E, F)$$

(and spelling it out, we have

$$\cdots \xrightarrow{d} \underline{\mathrm{Map}}_{\mathcal{C}}(E,F)^{i} \xrightarrow{d} \underline{\mathrm{Map}}_{\mathcal{C}}(E,F)^{i+1} \xrightarrow{d} \cdots$$

s.t. $d^2=0$)

- For $E \in \mathrm{Ob}(\mathcal{C})$, a morphism $k \longrightarrow \underline{\mathrm{Map}}_{\mathcal{C}}(E,E)$ (the <u>unit</u> of E)

- For $E, F, G \in \mathrm{Ob}(\mathcal{C})$, a morphism

$$\underline{\mathrm{Map}}_{\mathcal{C}}(E,F) \otimes_k \underline{\mathrm{Map}}_{\mathcal{C}}(F,G) \longrightarrow \underline{\mathrm{Map}}_{\mathcal{C}}(E,G)$$

& these "composition maps" are associative & unital <u>on the nose</u>.

<u>Rmk</u> A k-linear dg category is a category enriched over $Ch(k)$ (= sym monoidal category of cochain complexes over k)

<u>Examples</u>

1) B = dg algebra$/k$ $\rightsquigarrow$ dg category with one object $*$ & $B = \underline{\mathrm{Map}}_{\mathcal{C}}(*,*)$

2) $Ch(k)_{dg}$ = k-cochain cplxs as a dg category,

<u>obj</u> : $\mathrm{Ob}(Ch(k))$

<u>mor</u> : $\underline{\mathrm{Map}}_{Ch(k)_{dg}}(E,F) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{Ch(k)}(E, F[i])$

Exercise : $Z^0 \underline{\mathrm{Map}}_{Ch(k)_{dg}}(E, F) \cong \mathrm{Hom}_{Ch(k)}(E, F)$

3. X scheme$/_k$ $\rightsquigarrow$ dg cat $L_{qcoh}(X)^\circ$

objects : ~~cochain complexes of quasicoherent $\mathcal{O}_X$-modules~~

cochain complexes of $\mathcal{O}_X$-modules s.t. on each affine patch, it is given by a cochain cplx of modules

morphisms : $\mathcal{O}_X$-linear maps of any coh. degree

$\rightsquigarrow$ now localize $L_{qcoh}(X)^\circ$ along quasi-isomorphisms.

A dg category is a kind of higher category. In particular, there is a dg nerve construction producing a quasicategory

As a gloss:

0-simplices $N_{dg}(\mathcal{C})_0$ = objects of $\mathcal{C}$

1-simplices $N_{dg}(\mathcal{C})_1 = \{ f : E \to F, \text{ deg } 0 \text{ w/ } df = 0 \}$

2-simplices $N_{dg}(\mathcal{C})_2 = \left\{ \begin{tikzcd} & F \arrow[d, "g"] \\ E \arrow[ur, "f"] \arrow[r, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow z"] & G \end{tikzcd} \quad \begin{array}{l} f, g, h \text{ deg } 0 \\ z \text{ deg } -1 \text{ a} \\ \text{htpy from } h \text{ to } g \circ f \end{array} \right\}$

$\vdots$

Rmk a dg functor $\mathcal{C} \xrightarrow{f} \mathcal{C}'$ is a map of sets $Ob(f) : Ob(\mathcal{C}) \to Ob(\mathcal{C}')$ and for each pair $E, F \in Ob(\mathcal{C})$, a map of complexes

$$\underline{Map}_{\mathcal{C}}(E, F) \xrightarrow{f_{E,F}} \underline{Map}_{\mathcal{C}'}(fE, fF)$$

& these respect units & associativity on the nose.

Rmk: for $E \in \mathcal{C}$,

$$H^{-i}\,\underline{Map}_{\mathcal{C}}(E, E) =: Ext^{-i}_{\mathcal{C}}(E, E) \quad , \ i \geq 0$$

are interpreted as "higher automorphisms" of E

These prevent representing the moduli of objects in $\mathcal{C}$ by an ordinary scheme. //

Def A dg functor $\mathcal{C} \xrightarrow{f} \mathcal{C}'$ is a quasi-equivalence if ① $\forall E, F \in Ob(\mathcal{C})$, $\underline{Map}_{\mathcal{C}}(E, F) \xrightarrow{f_{E,F}} \underline{Map}_{\mathcal{C}'}(fE, fF)$ is a quasi-isomorphism

– the functor $H^0(f): H^0(\mathcal{C}) \to H^0(\mathcal{C}')$
is an equivalence of categories.

$$H^0(\mathcal{C}) = \begin{cases} \text{obj} = Ob(\mathcal{C}) \\ \text{mor} = H^0 Map_{\mathcal{C}}(E,F) \end{cases}$$

Ex: if $B \xrightarrow{\simeq} B'$ is a quasi-iso of dg algebras,
then their "dg categories" are
quasi-equivalent

Thm (Tabuada)
There is a model structure on $dgcat_k$
for which quasi-equivalence is
the weak equivalences

Def Let $dgcat_k$ be the ∞-category of
small k-linear dg categories.

Def $\mathcal{C} \in dgcat_k$, a $\mathcal{C}$-module is a
dg functor $M: \mathcal{C} \longrightarrow Ch(k)_{dg}$.
multi-object generalization of usual notion of modules

Def $\hat{\mathcal{C}}$ = dg-category of $\mathcal{C}^{op}$-dg-modules
but this is (typically) not small, so
it is "big"

Let $dgcat_k^{big}$ be the ∞-category of big k-linear, presentable dg categories

useful technical condition

basically nice localizations of dg categories like $\hat{\mathcal{C}}$

<u>Ex</u> $L_{qcoh}(X) \supseteq L_{perf}(X)$ = subcategory whose objects are perfect cplxs (locally are bounded cplxs of proj modules of finite rank)

$L_{qcoh}(X) \in dgcat_k^{big}$

$L_{perf}(X)$ essentially small

determined by shifts & retracts & cones of A, locally

$L_{qcoh}(X) \simeq \operatorname{Ind}(L_{perf}(X))$

completion under filtered colimits

§2 Moduli of branes

$\mathcal{C} = L_{qcoh}(X)$, X scheme$/k$

$$A \in CAlg_k, \quad L_{qcoh}(X \times \operatorname{Spec} A) \simeq \underbrace{L_{qcoh}(X) \otimes_{Mod_k^{\otimes}} Mod_A^{\otimes}}_{\substack{\text{tensor product} \\ \text{in } dgcat_k^{big}}}$$

<u>Def</u> $\mathcal{C} \in dgcat_k^{big}$

$$\mathcal{M}_{\mathcal{C}}^{b} : CAlg_k \longrightarrow dgcat$$

$$A \longmapsto (\mathcal{C} \otimes_{Mod_k} Mod_A)^{c}$$

← subcategory of compact objects

<u>N.B.</u> $\mathcal{C} \otimes_{Mod_k} Mod_A \simeq Mod_A(\mathcal{C}) =$ (∞-category of objects $E \in \mathcal{C}$ w/ action $A \otimes_k E \to E$ + associativity ...)

$\wr\!\wr$

$Fun^{L}(Mod_A, \mathcal{C})$

$$\mathcal{M}_{\mathcal{C}} := N_{dg}^{\simeq} \circ \mathcal{M}_{\mathcal{C}}^{b} : CAlg_k \longrightarrow \mathcal{S}$$

$$\mathcal{M}_{\mathcal{C}}(A) = N_{dg}(Mod_A(\mathcal{C})^{c})^{\simeq}$$

Thm (Toën-Vaquié)

If $\mathcal{C} \in dgcat_k^{big}$ is smooth & proper,
then $\mathcal{M}_{\mathcal{C}}$ is a locally geometric stack.

In particular, it has a cotangent complex.
We might hope then to extract a formal moduli problem, but this works more generally (not just for smooth & proper $\mathcal{C}$)

Def $E \in Ob(\mathcal{C})$, $ObjDef_E = (\mathcal{M}_{\mathcal{C}})^{\wedge}_{E}$ is the formal completion of $\mathcal{M}_{\mathcal{C}}$ at E.

$$ObjDef_E : CAlg_k^{sm} \longrightarrow \mathcal{S}$$

Fact: In general, it is not a formal moduli problem. But if E is connective with respect to a nice t-structure on $\mathcal{C}$, it is. (Lurie)

Guess the dg Lie algebra $\underline{\mathrm{End}}_{\mathcal{C}}(E)$
should describe deformations of E
(cf the case of a cochain cplx)

Thm (Lurie)
For E an object of a big dg category $\mathcal{C}$,
there is a natural map of functors

$$\mathrm{ObjDef}_E \longrightarrow \mathrm{ObjDef}^{\wedge}_E := \underbrace{\Psi(k \oplus \underline{\mathrm{End}}_{\mathcal{C}}(E))}_{E_1 \text{ formal moduli problems}}$$

from Alg^{sm}_k to $\mathcal{S}$. It is
universal among all maps from
ObjDef to E_1 FMPs. Moreover it induces
an iso on π_i for $i \geq 1$ & an
injection on π_0.

Rmk The difference can be understood as
between QCoh & IndCoh

$$\mathrm{ObjDef}^{\wedge}_E(A) = \mathrm{Map}_{\mathrm{Alg}_k}(\mathbb{D}^{(1)}(A), \underline{\mathrm{End}}(E)) \simeq \mathrm{LMod}_{\mathbb{D}^{(1)}}(A)_E$$

$$\mathrm{ObjDef}_E(A) = \mathrm{RMod}(A)_E \hookrightarrow \mathrm{RMod}^!_{\mathbb{A}}(A)_E$$